Quiz 8.2: Sample Answers

1. At noon, ship A is a distance $d = 130$ km west of ship B. Ship A is sailing east at a speed of 35km/h, ship B is sailing north at a speed of 25km/h. How fast is the distance between the ships changing at 3pm?

   Let P be the point where ship B is at noon. Let a be the distance of ship A from P, let b be the distance of ship B from P, and let c be the distance between the two ships. We know $\frac{db}{dt} = 25$ and $\frac{da}{dt} = -35$ (negative since it is sailing towards P). We wish to find $\frac{dc}{dt}$.

   By Pythagoras, $c^2 = a^2 + b^2$. Taking the derivative with respect to time t gives

   $$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$$

   Simplifying this gives

   $$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c}$$

   At 3pm, $a = 130 - 3(35) = 25$, $b = 25(3) = 75$, and using $c^2 = a^2 + b^2$ gives $c = \sqrt{6250}$. We can then substitute these values to get

   $$\frac{dc}{dt} = \frac{(25)(-35) + (75)(25)}{\sqrt{6250}} = 12.65$$

   Thus the distance between the ships is changing at a rate of 12.65 km/h at 3pm.

2. A street light is mounted at the top of a pole with a height of 12ft. A man 6ft tall walks away from the pole with a speed of 3ft/s along a straight line. How fast is the tip of his shadow moving when he is $x = 40$ft from the pole?

   Let x be the distance of the man from the pole, y the distance from the man to his shadow, and z the distance of the shadow from the pole. We know $\frac{dx}{dt} = 3$ and we want to find $\frac{dz}{dt}$.

By similar triangles, $\dfrac{y}{6} = \dfrac{z}{12}$. Thus $z = 2y$. Since $y = z - x$, we get $z = 2(z - x)$, which makes $z = 2z - 2x$, or $z = 2x$.

Taking the derivative of both sides with respect to time gives

$$\frac{dz}{dt} = 2\frac{dx}{dt}$$

Thus substituing $\frac{dx}{dt} = 3$ gives $\frac{dz}{dt} = 6$. Thus the tip of the shadow is moving at a rate of 6ft/s.

3. Gravel is being dumped from a conveyor belt at a rate of 35 cubic ft/m and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing at the time when the height is h = 8ft?

   Let V be the volume of gravel. We know $\frac{dV}{dt}$ and we wish to find $\frac{dh}{dt}$. Since the pile is in the shape of a cone, $V = \frac{\pi r^2 h}{3}$. However, we know that the diamater = 2r is always equal to the height, so $h = 2r$, giving $r = \frac{h}{2}$. Subsituting this into the volume formula gives

$$V = \frac{\pi h^3}{12}$$

   Taking the derivative with respect to time t gives

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

   Solving for the required quantity, we get

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{\pi(7)^2}(35) = 0.697$$

   Thus the height of the pile is increasing at a rate of 0.697ft/sec.

4. Water is leaking out of an inverted conical tank at a rate of 11 litres/min. At the same time water is being pumped into the tank at a constant rate. The tank has a height of 7m. Its diameter at the top is 4m. Find

the rate at which water is being pumped into the tank if the water level is rising at a rate of 21 cm/min when the height of the water in the tank is 3m.

Let V be the volume of water in the tank, h the height of the water, r the radius of the water, and x the rate at which water is flowing into the tank. We know $\frac{dh}{dt} = 0.21$, and we know $\frac{dv}{dt} = x - 11$, so we need to relate h and V to find x.

The volume of a cone gives $V = \frac{1}{3}\pi r^2 h$. Since we want to relate h and V, we need to eliminate r. Since the height of the tank is 7 and the radius at the top is 2, by similar triangles, we know that $\dfrac{r}{h} = \dfrac{2}{7}$. Thus $r = \frac{2}{7}h$. So we have the equation

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{2}{7}h\right)^2 h = \frac{4\pi}{3(49)}h^3$$

Taking the derivative with respect to time t, we get

$$\frac{dV}{dt} = \frac{4\pi}{49}h^2\frac{dh}{dt}$$

We can now subsitute our values. Note that since we wish to get our answer in litres, we have to convert the height measurements appropiately. Thus $h = 30$ and $\frac{dh}{dt} = 2.1$. Substituting, we get:

$$x - 11 = \frac{4\pi}{49}(30)^2(2.1)$$

Thus gives $x = 495.7$. Thus water is being pumped into the tank at a rate of 495.7 litres/min.